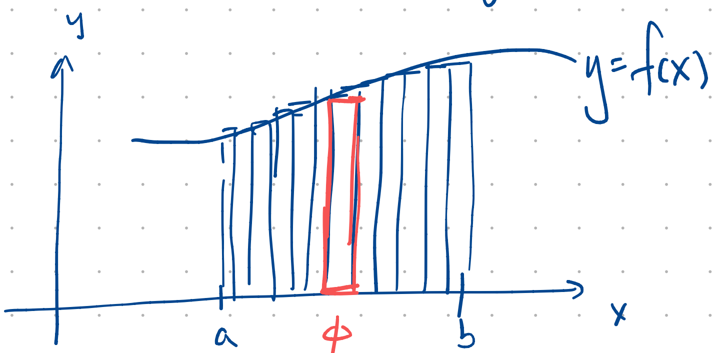
OVERVIEW OF §15.1

Let's begin by revisiting a bit of SVC.

If you want the area under $y=f(x)$:

1



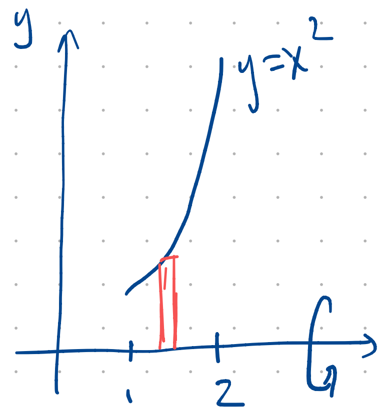
"Area $= f(x)\,dx$"

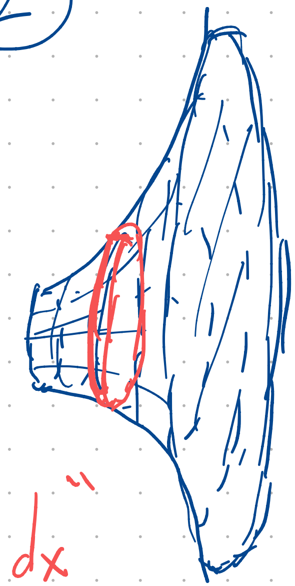
Total area = add them up and compute

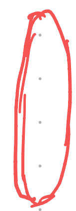
$$\int_a^b f(x)\,dx.$$

Also, you perhaps learned how to compute volumes of solids of revolution!

2





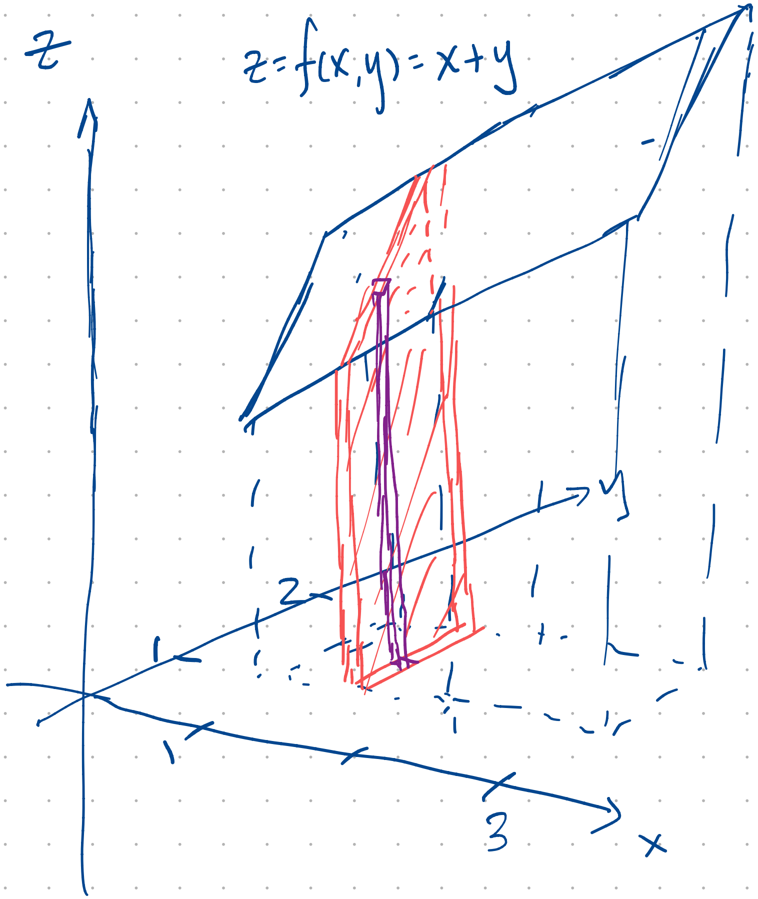


"Volume $= \pi y^2\,dx$"

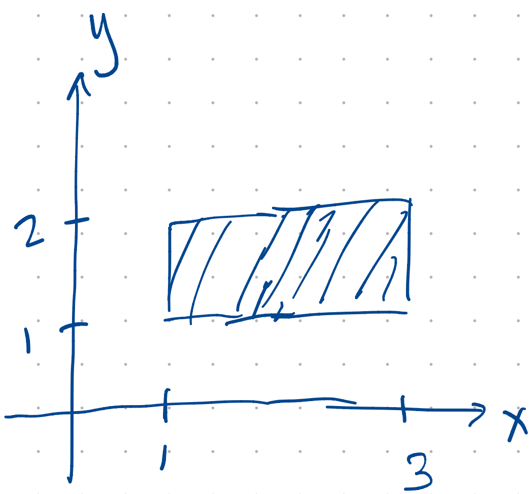
$$= \pi (x^2)^2\,dx$$

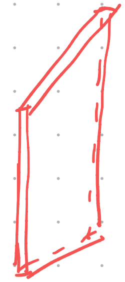
Total volume: add them up

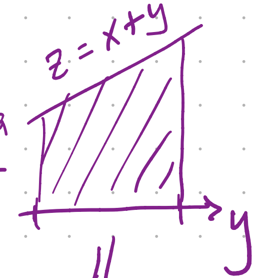
$$\int_1^2 \pi x^4\,dx.$$



View from above:



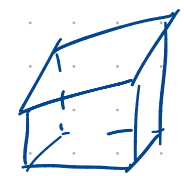


Volume = " (Area of ) dx "

||

$$\int_1^2 (x+y)\,dy$$

x is a constant along each slice!!

So total vol of  : add the slices up:

$$\int_1^3 \left(\int_1^2 (x+y)\,dy\right) dx$$

This is a combination of ideas from 1 and 2 on prev. page.

Namely (2) for the outer integral, (1) for inner.

Let's compute:

$$\int_1^3 \int_1^2 (x+y)\,dy\,dx = \int_1^3 \left(xy + \frac{1}{2}y^2\right)\Big|_{y=1}^{2} dx$$

$$= \int_1^3 (2x+2) - \left(x + \frac{1}{2}\right) dx$$

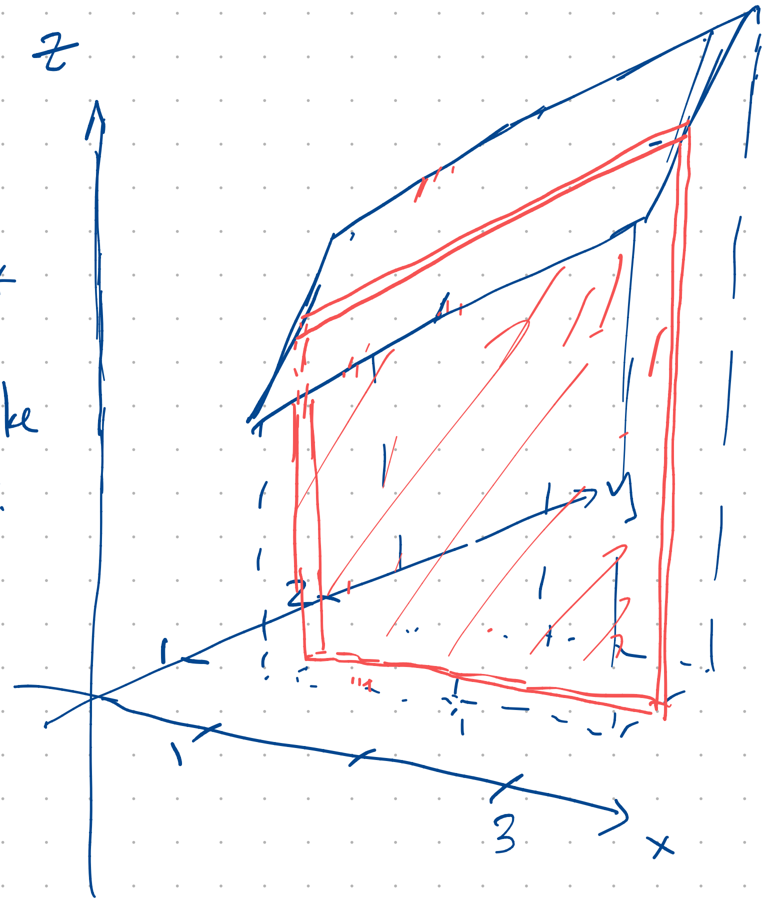
$$= \int_1^3 x + \frac{3}{2}\, dx$$

$$= \left(\frac{1}{2}x^2 + \frac{3}{2}x\right)\Big|_{x=1}^{3} = \left(\frac{9}{2} + \frac{9}{2}\right) - \left(\frac{1}{2} + \frac{3}{2}\right)$$

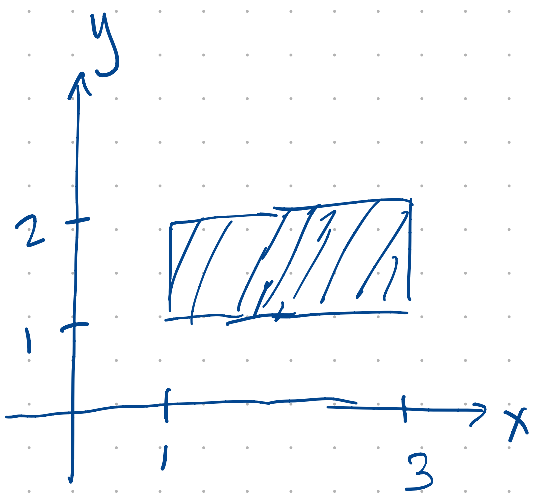
$$= \boxed{7}$$

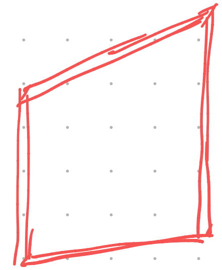
Observe: this depends only on x, just as how the disk slices in (2) depend only on x.

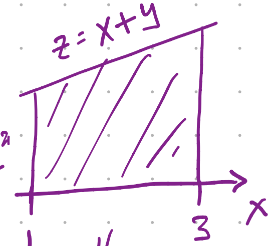
But I could also break up the region like this:



View from above:





Vol = "(Area of ) dy"

||

$$\int_1^3 (x+y)\,dx$$

y is a constant along each slice!!

Total volume: add up the slices

$$\int_1^2 \left(\int_1^3 (x+y)\,dx \right) dy$$

COMPARISON

$$\int_1^3 \int_1^2 (x+y)\, dy\, dx$$

$= \ldots\ldots$ (by calcs done previously)

$= \boxed{7}$

This agreement $(7=7)$ is a manifestation of Fubini's Thm (15.1.10).

"Fubini is to integration as Clairaut is to differentiation."

$$\int_1^2 \int_1^3 (x+y)\, dx\, dy$$

$$= \int_1^2 \left(\tfrac{1}{2}x^2 + xy\right)\Big|_{x=1}^{3} dy$$

$$= \int_1^2 \left(\tfrac{9}{2} + 3y\right) - \left(\tfrac{1}{2} + y\right) dy$$

$$= \int_1^2 4 + 2y\, dy = \left(4y + y^2\right)\Big|_{y=1}^{2}$$

$$= (8+4) - (4+1) = \boxed{7}$$

Intuitive explanation of Fubini: still adding up the same stuff, but in different order.

1	7	-2	3
0	4	8	-5
3	2	1	0

$$\begin{aligned}&(1+0+3)\\&+(7+4+2)\\&+(-2+8+1)\\&+(3+(-5)+0)\end{aligned} = \begin{aligned}&(1+7+(-2)+3)\\&+(0+4+8+(-5))\\&+(3+2+1+0)\end{aligned}$$

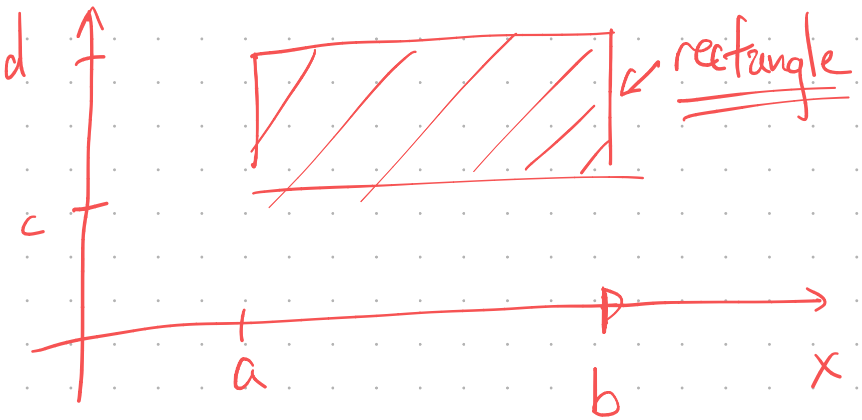
⚠ Something you may have seen in Math 1A/B: If a series is not absolutely convergent, then rearranging can actually result in different values.

So despite the illustration to the left, Fubini's Thm is actually a very nontrivial result.

⚠ All this is still in the context of §15.1 where

$$\int_a^b \int_c^d f(x,y)\, dy\, dx$$

a, b, c, d are all constant, i.e.,



If the region is not a rectangle, can still do both dydx and dxdy orders, but the bounds are more involved to interchange (§15.2, we'll discuss this next time).